Calculus II - Day 9

Prof. Chris Coscia, Fall 2024
Notes by Daniel Siegel

2 October 2024

Power Series

Goals for today

- Define power series and determine their radius and interval of convergence.
- Use the power series of $\frac{1}{1-x}$ to find power series for similar functions using substitution or multiplication.
- Differentiate/integrate "term-by-term" to obtain new power series and estimation.

Midterm logistics

- 12:00-1:15 on Wednesday, October 16^{th} in Cohen Auditorium (Aidekman)
- Review materials posted to Canvas.
- Bring one 8.5x11 in. (or smaller) sheet of notes (both sides okay).
- Can be written or typed but *must be a physical sheet of paper* at the exam.
- In-class review Wednesday, October 9^{th} – answer the survey to suggest topics.

Recall: Geometric Series

$$\sum_{k=0}^{\infty} ar^k \text{ converges to } \frac{a}{1-r} \text{ if } -1 < r < 1$$

We can think about this as a function:

$$f(x) = \frac{1}{1-x}$$

which can be written as a series when $-1 < x < 1$:

$$f(x) = \sum_{k=0}^{\infty} x^k$$

This is a **power series**.

Definition: A ***power series*** is a function of the form:

$$f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$$

Where a is the <u>*center*</u> of the power series, and the c_k's are real numbers that depend on k, called coefficients. x is a variable.

Where do power series converge?

Example:

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \dots$$

This is a power series centered at $a = 0$, and $c_k = 1$ for every k.
This is a function – let's try plugging in a number.
If $x = \frac{1}{3}$:

$$\sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k = 1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \dots = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2}$$

If we try to plug in $x = -2$,

$$\sum_{k=0}^{\infty} (-2)^k = 1 - 2 + 4 - 8 + 16 - \dots \quad \text{(diverges)}$$

The interval of convergence of this series is $I = (-1, 1)$.
The radius of convergence is $R = 1$.
Example: For what values of x does the series $\sum_{k=0}^{\infty} k!x^k$ converge?
Center: $a = 0$, coefficients $c_k = k!$.
Use the Ratio Test to determine where this series converges:

$$r = \lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \to \infty} \left| \frac{(k+1)!x^{k+1}}{k!x^k} \right|$$

Simplifying:

$$= \lim_{k \to \infty} |(k+1)x| = |x| \lim_{k \to \infty} (k+1)$$

This expression diverges if $x \neq 0$, but if $x = 0$, $r = 0 < 1$, so the series converges only at $x = 0$.
Interval of convergence: $\{0\}$ $\leftarrow$ set that only contains 0
Radius of convergence: $R = 0$

Example: For what values of x does the series

$$\sum_{k=1}^{\infty} \frac{(x-2)^k}{k^2}$$

converge?
Center: $a = 2$, coefficients $c_k = \frac{1}{k^2}$.
Use the Ratio Test:

$$r = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{(x-2)^{k+1} \cdot k^2}{(k+1)^2 \cdot (x-2)^k} \right|$$

Simplifying:

$$= \lim_{k\to\infty} \left| \frac{(x-2) \cdot k^2}{(k+1)^2} \right| = |x-2| \lim_{k\to\infty} \left(\frac{k^2}{(k+1)^2} \right) = |x-2|$$

The series converges when $r = |x-2| < 1$, so:

$$-1 < (x-2) < 1 \quad \Rightarrow \quad 1 < x < 3$$

The series diverges when $r = |x-2| > 1$, which occurs when $x < 1$ or $x > 3$.
Inconclusive when $x = 1$ or $x = 3$. Let's check these cases separately:

$$x = 3:$$

$$\sum_{k=1}^{\infty} \frac{(x-2)^k}{k^2} = \sum_{k=1}^{\infty} \frac{(3-2)^k}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

This converges by the p-test since $p = 2 > 1$.

$$x = 1:$$

$$\sum_{k=1}^{\infty} \frac{(1-2)^k}{k^2} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$$

This converges as well, by the Alternating Series Test (AST).
Radius of convergence: $R = 1$
Interval of convergence: $I = [1, 3]$

Example: For what values of x does the series

$$\sum_{k=1}^{\infty} \frac{(x+7)^k}{k^k}$$

converge?
Center: $a = -7$.

Root Test:

$$\rho = \lim_{k\to\infty} |a_k|^{1/k} = \lim_{k\to\infty} \left| \frac{(x+7)^k}{k^k} \right|^{1/k}$$

Simplifying:

$$= \lim_{k\to\infty} \left| \frac{x+7}{k} \right| = |x+7| \cdot \lim_{k\to\infty} \left(\frac{1}{k} \right) = 0 \quad \text{for all } x$$

$\Rightarrow$ The series converges for all x.
Radius of convergence: $R = \infty$
Interval of convergence: $I = (-\infty, \infty)$ (all real numbers).

Let $\sum c_k(x-a)^k$ be a power series centered at $x = a$.

There are three things that can happen when we use the Ratio or Root Test to find the interval of convergence:

1. The series converges only at its center, $x = a$.

$$R_{\text{OC}} : R = 0$$

$$I_{\text{OC}} : I = \{a\}$$

2. The series converges for all x.

$$R_{\text{OC}} : R = \infty$$

$$I_{\text{OC}} : I = (-\infty, \infty) = \mathbb{R}$$

3. There is a positive number R such that:
   - The series converges when $|x-a| < R \Rightarrow a - R < x < a + R$.
   - The series diverges when $|x-a| > R \Rightarrow x > a + R$ or $x < a - R$.
   - We must individually check the endpoints to see whether or not the series converges at $x = a - R$ and $x = a + R$.

$$R_{\text{OC}} : R$$

$$I_{\text{OC}} : (a-R, a+R),\ [a-R, a+R],\ (a-R, a+R],\ [a-R, a+R)$$

Example: Find the interval of convergence of the power series

$$\sum_{k=1}^{\infty} \frac{(x-1)^k}{2\sqrt[k]{k}}$$

Center: $a = 1$. Use the Ratio Test:

$$r = \lim_{k\to\infty} \left| \frac{(x-1)^{k+1}}{2\sqrt[k+1]{k+1}} \cdot \frac{2\sqrt[k]{k}}{(x-1)^k} \right|$$

Simplifying:

$$= \lim_{k\to\infty} \left| \frac{(x-1)\cdot \sqrt[k]{k}}{2\cdot \sqrt[k+1]{k+1}} \right| = \frac{|x-1|}{2} \lim_{k\to\infty} \frac{\sqrt[k]{k}}{\sqrt[k+1]{k+1}}$$

As $k \to \infty$, $\frac{\sqrt[k]{k}}{\sqrt[k+1]{k+1}} \to 1$, so:

$$r = \frac{|x-1|}{2}$$

For convergence:

$$\frac{|x-1|}{2} < 1 \quad \Rightarrow \quad |x-1| < 2$$

Thus, the radius of convergence is $R = 2$.

* The series converges when $-2 < x - 1 < 2$, which simplifies to $-1 < x < 3$. * The series is inconclusive at $x = -1$ and $x = 3$. * The series diverges otherwise.

Radius of Convergence: $R = 2$

Case 1: $x = 3$

$$\sum_{k=1}^{\infty} \frac{(3-1)^k}{2\sqrt[k]{k}} = \sum_{k=1}^{\infty} \frac{2^k}{2\sqrt[k]{k}} = \sum_{k=1}^{\infty} \frac{2^{k-1}}{\sqrt[k]{k}}$$

The series diverges at $x = 3$.

Case 2: $x = -1$

$$\sum_{k=1}^{\infty} \frac{(-1-1)^k}{2\sqrt[k]{k}} = \sum_{k=1}^{\infty} \frac{(-2)^k}{2\sqrt[k]{k}} = \sum_{k=1}^{\infty} \frac{(-2)^k}{2^k \cdot \sqrt[k]{k}} = \sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt[k]{k}}$$

The series converges at $x = -1$ by the Alternating Series Test (AST).

Final conclusion:

Radius of Convergence: $R = 2$

Interval of Convergence: $I = [-1, 3)$

Example: $x^3 + x^4 + x^5 + x^6 + \cdots = \sum_{k=3}^{\infty} x^k$

Does this converge to a function we can write down explicitly?

This is the same as:

$$\sum_{k=0}^{\infty} x^{k+3} = \sum_{k=0}^{\infty} x^3 x^k = x^3 \sum_{k=0}^{\infty} x^k$$

$$\boxed{\sum_{k=0}^{\infty} x^k} \quad \leftarrow \quad \text{This converges to} \quad \frac{1}{1-x} \quad \text{when} \quad -1 < x < 1$$

Thus, the original series becomes:

$$x^3 \cdot \frac{1}{1-x} = \frac{x^3}{1-x}$$

Radius and Interval of Convergence:

$$R = 1, \quad I = (-1, 1)$$

Example: Express $\frac{1}{1+2x^2}$ as a power series. Where does it converge?
We know:

$$\frac{1}{1-u} = \sum_{k=0}^{\infty} u^k$$

Set $u = -2x^2$:

$$\frac{1}{1+2x^2} = \sum_{k=0}^{\infty} (-2x^2)^k = \sum_{k=0}^{\infty} (-1)^k 2^k x^{2k}$$

Expanding this gives:

$$= 1 - 2x^2 + 4x^4 - 8x^6 + \dots$$

Radius and Interval of Convergence:
The series converges when $|u| = |-2x^2| < 1$, i.e.

$$|x^2| < \frac{1}{2} \quad \Rightarrow \quad |x| < \frac{1}{\sqrt{2}}$$

Thus, the radius of convergence is:

$$R = \frac{1}{\sqrt{2}}, \quad I = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

This converges where $|u| < 1$:

$$|-2x^2| < 1$$

Simplifying:

$$x^2 < \frac{1}{2} \quad \Rightarrow \quad -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$$

Thus, the interval of convergence is:

$$I = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$